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A Priori Bounds for Periodic Solutions of a Delay Rayleigh Equation

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Abstract—*A priori* bounds are established for periodic solutions of a Rayleigh equation with delay. By means of these bounds, an existence theorem for periodic solutions can be obtained by means of Mawhin's continuation theorem.

Keywords—Rayleigh equation, Delay, Periodic solution.

In [1], continuation theorems are introduced and applied to the existence of solutions of differential equations. In particular, a specific example is given in [1, p. 99] (see also [2, p. 175]) on how periodic solutions can be obtained by means of these theorems for the differential equation

$$x''(t) + f(x'(t)) + h(t, x(t)) = 0,$$

where f is a real continuous function defined on R such that $f(0) = 0$, h is a real continuous function defined on $R \times R$, 2π-periodic in t and $h(t,x)x < 0$ for $|x| \geq r$ and $t \in [0, 2\pi]$. In the course of derivations, it is realized that once appropriate *a priori* bounds for the 2π-periodic solutions of the equation

$$x''(t) + \lambda f(x'(t)) + \lambda h(t, x(t)) = 0$$

are known for each $\lambda \in (0,1)$, then standard procedures will allow these theorems to imply existence of periodic solutions to the original equation.

In this note, we will be concerned with a similar equation with an additional delay

$$x''(t) + \lambda f(x'(t)) + \lambda g(x(t - \tau(t))) = \lambda p(t), \qquad \lambda \in (0,1), \tag{1}$$

where f, g, p, and τ are real continuous functions defined on R such that $f(0) = 0$, τ and p are periodic with period 2π, and

$$\int_0^{2\pi} p(t)\, dt = 0.$$

We will establish *a priori* bounds for solutions of this equation under relatively simple conditions on f and g. Once these bounds are obtained, existence of periodic solutions can be demonstrated for the equation

$$x''(t) + f(x'(t)) + g(x(t - \tau(t))) = p(t). \tag{2}$$

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We remark that there are a number of studies which are concerned with the existence of periodic solutions of differential equations by means of establishing *a priori* bounds (see, e.g., [3]). Other related results can be found in [4–6].

THEOREM 1. *Suppose there are positive constants K, D, and M such that*

(i) $|f(x)| \le K$ *for* $x \in R$,
(ii) $xg(x) > 0$ *and* $|g(x)| > K$ *for* $|x| \ge D$, *and*
(iii) $g(x) \ge -M$ *for* $x \le -D$.

Then for any 2π-periodic solution $x = x(t)$ of (1), $|x'(t)| \le D_2$ and $|x(t)| \le D + 2\pi D_2$ for some positive number D_2.

PROOF. Let $x = x(t)$ be a 2π-periodic solution of the equation (1). Since $x(0) = x(2\pi)$, there is some $t_0 \in [0, 2\pi]$ such that $x'(t_0) = 0$. In view of (1), we see that for any $t \in [0, 2\pi]$,

$$\begin{aligned} |x'(t)| &= \left| \int_{t_0}^{t} x''(s)\, ds \right| \le \int_0^{2\pi} |x''(s)|\, ds \\ &\le \lambda \int_0^{2\pi} |f(x'(s))|\, ds + \lambda \int_0^{2\pi} |g(x(s-\tau(s)))|\, ds + \lambda \int_0^{2\pi} |p(s)|\, ds \\ &\le 2\pi K + \int_0^{2\pi} |g(x(s-\tau(s)))|\, ds + 2\pi \max_{0\le s\le 2\pi} |p(s)|. \end{aligned} \tag{3}$$

We assert that

$$\int_0^{2\pi} |g(x(s-\tau(s)))|\, ds \le 2\pi K + 4\pi D_1 \tag{4}$$

for some positive number D_1. Indeed, integrating equation (1) from 0 to 2π, and noting condition (i), we see that

$$\begin{aligned} \int_0^{2\pi} \{g(x(t-\tau(t)) - K\}\, dt &\le \int_0^{2\pi} \{g(x(t-\tau(t)) - |f(x'(t))|\}\, dt \\ &\le \int_0^{2\pi} \{f(x'(t)) + g(x(t-\tau(t)))\}\, dt = 0. \end{aligned} \tag{5}$$

Thus letting

$$E_1 = \{t \in [0, 2\pi] \mid x(t-\tau(t)) > D\}, \qquad E_2 = [0, 2\pi] \backslash E_1,$$

we have

$$\int_{E_2} |g(x(t-\tau(t)))|\, dt \le 2\pi \max\left\{ M, \sup_{|x|\le D} |g(x)| \right\},$$

and

$$\begin{aligned} &\int_{E_1} \{|g(x(t-\tau(t)))| - K\}\, dt \le \int_{E_1} |g(x(t-\tau(t))) - K|\, dt \\ &= \int_{E_1} \{g(x(t-\tau(t))) - K\}\, dt \le -\int_{E_2} \{g(x(t-\tau(t))) - K\}\, dt \\ &\le \int_{E_2} |g(x(t-\tau(t)))|\, dt + \int_{E_2} K\, dt. \end{aligned}$$

Therefore,

$$\int_0^{2\pi} |g(x(t-\tau(t)))|\, dt \le 2\pi K + 4\pi \max\left\{ M, \sup_{|x|\le D} |g(x)| \right\}$$

as required. Combining (3) and (4), we see that

$$|x'(t)| \le D_2, \qquad t \in [0, 2\pi] \tag{6}$$

for some positive number D_2. Next, note that the last equality in (5) implies

$$f(x'(t_1)) + g(x(t_1 - \tau(t_1))) = 0$$

for some t_1 in $[0, 2\pi]$. Thus in view of condition (i),

$$|g(x(t_1 - \tau(t_1)))| = |f(x'(t_1))| \le K,$$

and in view of (ii),

$$|x(t_1 - \tau(t_1))| < D.$$

Since $x(t)$ is 2π-periodic, we may infer that $|x(t_2)| < D$ for some t_2 in $[0, 2\pi]$. Finally, we see that

$$\begin{aligned} |x(t)| &= \left| x(t_2) + \int_{t_2}^{t} x'(t)\,dt \right| \le D + \int_0^{2\pi} |x'(t)|\,dt \\ &\le D + 2\pi D_2, \qquad t \in [0, 2\pi]. \end{aligned} \tag{7}$$

The proof is complete.

By means of the *a priori* bounds just obtained, we may follow the standard procedures as explained in various places of [1] and the continuation theorem in [1, p. 40] to show the existence of a periodic solution of (2). For the sake of completeness, a brief sketch is included as follows.

Let X be the Banach space of all continuous differentiable functions of the form $x = x(t)$, defined on R such that $x(t + 2\pi) = x(t)$ for all t, and endowed with the norm $\|x\|_1 = \max_{0 \le t \le 2\pi} \{|x(t)|, |x'(t)|\}$. Also let Y be the Banach space of all continuous functions of the form $y = y(t)$, defined on R such that $y(t + 2\pi) = y(t)$ for all t, and endowed with the norm $\|y\|_0 = \max_{0 \le t \le 2\pi} |y(t)|$. Finally let Ω be the subspace of X containing functions of the form $x = x(t)$, such that $|x(t)| < \bar{D}$ and $|x'(t)| < \bar{D}$, where $\bar{D}$ is a fixed number greater than $D + 2\pi D_2$. Now let $L : X \cap C^{(2)}(R, R) \to Y$ be the differential operator defined by $(Lx)(t) = x''(t)$ for $t \in R$, and let $N : X \to Y$ be defined by

$$(Nx)(t) = -f(x'(t)) - g(x(t - \tau(t))) + p(t), \qquad t \in R.$$

Let $\operatorname{Im} L$ and $\operatorname{Ker} L$ be, respectively, the image and kernel of the operator L. Clearly, $\operatorname{Ker} L = R$. Furthermore, if we define the projections $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y/\operatorname{Im} L$ by

$$(Px)(t) = \frac{1}{2\pi} \int_0^{2\pi} x(t)\,dt, \qquad t \in R$$

and

$$(Qy)(t) = \frac{1}{2\pi} \int_0^{2\pi} y(t)\,dt, \qquad t \in R,$$

respectively, then $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, the operator L is a Fredholm operator with index zero, and the operator N is L-compact on the closure $\bar{\Omega}$ of Ω (see, e.g., [1, p. 176]).

For any $\lambda \in (0, 1)$ and any $x = x(t)$ in the domain of L which also belongs to $\partial\Omega$, we must have $Lx \ne \lambda Nx$. For otherwise in view of (6) and (7), we see that x belongs to the interior of Ω, which is contrary to the assumption that $x \in \partial\Omega$. Next, note that a function $x = x(t)$ in the intersection of $\operatorname{Ker} L$ and $\partial\Omega$ must be the constant functions $x(t) \equiv \bar{D}$ or $x(t) \equiv -\bar{D}$. Hence

$$(QN)(x) = \frac{1}{2\pi} \int_0^{2\pi} \left(-f(x'(t)) - g(x(t - \tau(t))) + p(t) \right) dt = -g\left(\pm\bar{D}\right) \ne 0.$$

Finally, consider the mapping

$$H(x, s) = sx + (1 - s)g(x), \qquad 0 \le s \le 1.$$

Since for every $s \in [0,1]$ and x in the intersection of $\operatorname{Ker} L$ and $\partial\Omega$, we have

$$xH(x,s) = sx^2 + (1-s)xg(x) > 0,$$

thus $H(x,s)$ is a homotopy. This shows that

$$\begin{aligned}\deg\{QNx, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{-g(x), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{-x, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{-x, \Omega \cap R, 0\} \neq 0.\end{aligned}$$

We have thus verified all the assumptions of the continuation theorem [1, p. 40]. Under the assumptions in Theorem 1, equation (2) thus has a 2π-periodic solution.

As an example, consider the equation

$$x''(t) + \exp\left\{-\left(x'(t)\right)^2\right\} + \arctan(x(t-\pi)) = \sin t + 1.$$

Take $f(u) = \exp\{-u^2\} - 1$, $g(u) = \arctan u$ and $p(u) = \sin u$, and $\tau = \pi$. It is then easy to verify that all the assumptions in Theorem 1 are satisfied with $K = 1$, $D > \pi/4$, and $M = \pi/2$. Thus this equation has a 2π-periodic solution.

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